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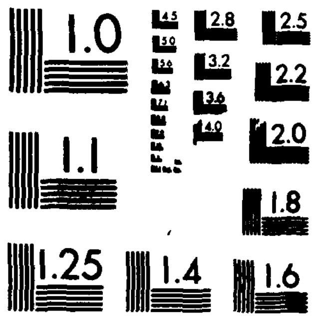

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The main results discussed in this report include;
(a) The modelling of rough surfaces via chi-squared random fields, and a discussion of the analytic properties of chi-squared processes and fields.
(b) The distribution of the supremum of Gaussian random fields and other more general Gaussian processes, centering on tail distributions.
(c) The relationship of (b) to the study of empirical processes and other problems related to empirical processses and measures.
(d) The relationship between Gaussian and non-Gaussian generalised processes and Markov processes via central and other limit theorems for functionals of Markov paths.

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THEORY AND APPLICATIONS OF RANDOM FIELDS

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Final Report, 1 October 1985 - 30 September 1986

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CONTENTS

1. INTRODUCTION

In the original research proposal that is being reported on here there were essentially four distinct, albeit related, projects. This report, for ease of writing, reading, and evaluation is built around these four projects in the following fashion.

For each project I have included a brief recapitulation of what was presented in the original proposal. (A reader who is familiar with, and still remembers the details of, the original proposal can bypass the recapitulation.) Following this is, in each of the four cases, a report on the progress made towards realizing the goals of the proposal. The reports are generally rather brief, since they merely summarize results already presented in research papers to which the reader can turn for more details.

Following each report is a brief comment on further avenues of research (if any) opened up by the work done to date.

At the end of the reports is a list of research papers that resulted from the research described, as well as a list of conferences attended and visits made to American universities under the auspices and financing of the grant.

For the sake of completeness, we commence by including some general background material on random fields.

2. SOME BACKGROUND ON RANDOM FIELDS

Random fields are simply stochastic processes, $X(t)$, whose "time" parameter, t, varies over some rather general space rather than over the more common real line. The simplest of these occur when the parameter space is some multi-dimensional Euclidean space, and it is these fields that will be at the centre of our study. Of these, the most basic arise when the parameter space is the two-dimensional plane, so that we are dealing with some kind of random surface. When the parameter space is three-dimensional then we have a field (such as ore concentration in a geological site) that varies over space, while when the dimension increases to four we are generally dealing with space-time problems.

More complicated examples of random fields arise as the parameter space becomes more esoteric. Typical examples are parameter spaces of classes of sets, such as arise in the statistical theory of multi-variate Kolmogorov-Smirnov tests and set-indexed empirical processes. Another common example is provided by fields indexed by families of functions. While the latter arise in the theory of empirical processes, they are much more well known via their appearance in Quantum Field Theory in Mathematical Physics. There they appear, among other guises, as continuum limits of such well known discrete parameter random fields as the Ising model of Statistical Mechanics.

For the moment, however, let us consider the simple setting of continuous parameter random fields defined on a Euclidean space. The theory of these fields is now quite substantial, with four separate monographs on various aspects of the subject having appeared in the past four years. (Adler (1981a), Rozanov (1982), Vanmarcke (1983) and Yadrenko (1983).) Roughly speaking, the theory breaks quite naturally into two quite distinct parts.

In the first case, we assume that the sample functions (realisations) of the random field satisfy certain basic regularity conditions, such as continuity, differentiability, etc.. It is then possible to study problems such as the structure of the field in the neighbourhood of extrema, and the rate at which the field "crosses" (a term which requires careful definition) various levels. These problems turn out to be very important in the application of random fields to the study of rough surfaces, as discussed in the following section.

The second class of problems in the study of continuous parameter random fields arises when the regularity conditions mentioned above are not imposed. Conventionally, one then studies such sample path properties as the (Hausdorff) dimension of various random sets generated by the field. While these fields, despite their somewhat esoteric properties, are both theoretically interesting and of applied importance, (as the current theory of fractal geometry due to Mandelbrot (1982) and his colleagues has shown beyond any shadow of doubt), they are only of peripheral concern to the main thrust of the current project.

Although, as just noted, the theory of continuous parameter random fields is well developed, it is important to note that in one respect at least it is still very restricted. This is a consequence of the fact that throughout the literature, both theoretical and applied, there is an almost universal assumption of normality. This is an assumption that has a substantial simplifying affect on the mathematics of random fields, but is undesirable for two quite distinct reasons. The first, which comes from purely practical considerations, is that real life fields to which one might like to apply the theory are very often non-Gaussian. For example, the rough metallic surfaces described in the following section are known to be highly non-Gaussian (Adler (1981b)). Assuming, incorrectly, that they are Gaussian leads to the development of a theory of surface structure that invariably fails to tie in with experiment. The second difficulty with the Gaussian assumption is that it hamstrings the Mathematician by limiting the phenomena available for his investigation to that case only.

Of the four reports that follow, two are intimately concerned with non-Gaussian processes. The first involves the development and study of a model that can often be used in place of a Gaussian one without too great an increase in the level of difficulty of the mathematics. The fourth is related to the construction of non-Gaussian (and Gaussian) generalised processes via the sample paths of Markov processes. The remaining two reports are concerned primarily with Gaussian, or closely related, processes.

Overall, the common thread that runs through the project is the extension of both the theory and applications of random fields, with the aim of increasing our understanding of the Gaussian situation while at the same time attempting to extend our horizons beyond it.

3(a) ROUGH SURFACES AND CHI-SQUARED PROCESSES

It is now a well established fact that all surfaces used in engineering practice are rough when judged by the standards of molecular dimensions. This fact has played a major role in the development of *Tribology*, a science that, among other problems, is concerned with the nature of contact between two surfaces under load and its relationship to problems such as wear, friction, and the conduction of heat and electricity between two surfaces in contact.

Because of the difficulties inherent in observing what happens when two surfaces are in actual physical contact Tribology has made substantial use of mathematical models. The basic idea underlying this has been to develop models of surface structure (at the microscopic level) and then apply these together with, say, a theory of surface deformation, to predict observable (macroscopic) phenomena. Although there has been an enormous amount of activity in this area over the past twenty years (see Thomas (1982) for a recent exhaustive survey) there is still very often disconcerting disagreement between theory and practice. This is despite the fact that very sophisticated random field models have been used for the rough surfaces.

The reason for this is very simple. Almost without exception, rough surfaces have been modeled as Gaussian fields, when, in fact, they are highly non-Gaussian. This point was emphasised in Adler and Firman (1981), following an analysis of both old and new rough surface data. Consequently, irregardless of the sophistication of the model, it is not surprising that the current models fail to yield a theory that squares with practice.

It was precisely this problem that initiated the current study of chi-squared (χ^2) processes and fields. Chi-squared processes can be easily defined by writing

$$X(t) = \Sigma_{i=1}^{n} [Y_i(t)]^2,$$

where the Y_i are a sequence of independent Gaussian processes

This simple trick yields a family of fields that are at the same time substantially different to Gaussian fields in their sample path behaviour and yet mathematically close enough to their Gaussian parents to be analytically tractable. Furthermore, it yields a family of fields that turn out to model rough surfaces very closely, and to generate a theory that yields results akin to those observed in the laboratory (c.f. Adler (1981b)).

It was from this background that it was decided, some three to four years ago, that a systematic study of χ^2 processes and fields be undertaken. This study was reasonably successful, and a reasonably full picture of the behaviour of χ^2 processes and fields is now available. This work has been written up in detail in the three joint papers with Michael Aronowich listed in the bibliography, the last of these having been prepared during the tenure of this year's grant. (Aronowich was a graduate student supported by the Technion over the past four years to work on projects related to random fields.)

Without going into detail here, let it suffice to say that we now know as much about χ^2 processes and fields as is known about Gaussian fields, at least insofar as their application to modelling problems is concerned. (There certainly remain many theorems of an abstract nature to be established, but our aims in this particular piece of research have always been primarily applied in nature.)

We plan in the future to commence a project of exploitation of the results we now have in problems of surface science. This turned out to be an essentially impossible task during 1985/86 while I was on sabbatical, and constantly moving around. Hopefully, however, it is a task that will be started in earnest now that I am back at the Technion with its environment of engineering science and the sort of permanent facilities (data files, familiar computing environment, etc.) that I did not have last year.

3(b) MAXIMA OF GAUSSIAN FIELDS

A simple sounding, but infamously difficult problem is the following: Let $X(t)$ be a stationary, Gaussian random process defined on some interval on the real line. What is the distribution of $\sup_t X(t)$?

The answer to this is known for only five covariance functions. A general theory, applicable to almost all Gaussian processes on the line, is available only if one is prepared to settle for approximations to tail probabilites of the form $P(\sup_t X(t) > \lambda)$ for large values of λ. (For a full telling of the theory of this, see Leadbetter, *et. al.* (1983).) The problem becomes even more complicated when one moves to random fields, and allows the parameter t to vary either over a subset of some Euclidean space or over some even more complex space. There, very little is known, even if one restricts attention to supposedly simple fields, such as the so called *Brownian sheet*, the natural generalisation to R^k, $k>1$, of the Brownian motion on the line.

Good, general, asymptotic bounds are, however, now known. For example, for the *pinned F-sheet*, W_F, of empirical process theory it was shown in Adler and Brown (1986), essentially solving a twenty year old problem of Kiefer's, that there exist constants $c_k(F)$ and C_k (independant of F) such that for all λ

$$c_k \lambda^{2(k-1)} e^{-2\lambda^2} < P\left(\sup_{t\in[0,1]^k} W_F(t) > \lambda \right) < C_k \lambda^{2(k-1)} e^{-2\lambda^2}.$$

(Here W_F is the Gaussian process on the k-dimensional unit cube with zero mean and covariance function $F(t_\wedge s) - F(t).F(s)$.)

It turns out that the above results, wh[illegible] theory of empirical processes we shall describe [illegible] section, can be extended as follows. If X(t) is a [illegible] Gaussian process, and T some general parameter set (e.g. [illegible] sets in the plane, or, more simply, just points in some subset [illegible] the plane) then there is a function $p(\lambda)$ and constants c and C (perhaps dependent on the process) such that under quite general side conditions

$$c.p(\lambda).e^{-\lambda^2/2\sigma^2} < P(\sup_{t \in T} X(t) > \lambda) < C.p(\lambda).e^{-\lambda^2/2\sigma^2}.$$

Here $\sigma^2 = \sup\{E[X^2(t)], t \in T\}$, and the form of p depends on the covariance function of X and the size (measured in terms of metric entropy) of the parameter set T. For example, for simple parameter sets such as points in R^k $p(\lambda)$ is usually a polynomial of order related to k. For larger, more complicated parameter sets, p may turn out itself to be an exponential of the form $\exp(\text{const}.\lambda^\alpha)$, where of necessity $\alpha<1$.

The establishment of the existence of such an extension was one of the projects suggested in our proposal a year and a half ago. At that stage we had a number of results (Adler and Samorodnitsky (1985)) pertaining to primarily polynomial forms of the function $p(\lambda)$ above. Since then, under the auspices of the current grant and further supported by general Technion funds Gennady Samorodnitsky has substantially sharpened our earlier joint results for the polynomial case and furthermore extended them to the case of exponential p. (c.f. Samorodnitsky 1986a,b).

It is my intention to continue with the study of the distribution of Gaussian extrema in two related connections. Firstly, despite the power of the results described above, when it comes to wanting to know something very precise for a very specific process, good results are still few and far between. Now, however, we have a good set of tools available, and so it would seem to be an opportune time to tackle the problem of precise estimates for specific processes. This is primarily an analytical, mathematical, problem, although computer simulation of extrema distributions is useful in identifying how good (or bad) are theoretically derived bounds.

The second problem is related to that of multivariate empirical processes and multivariate Kolmogorov-Smirnov (KS) statistics.

3(c) EMPIRICAL PROCESSES AND MULTIVARIATE KOLMOGOROV-SMIRNOV

This theory arises in problems of testing whether a specific, hypothesised, multivariate distribution is consistent with a particular set of data, and of testing for independence between sets of variables. A good recent review is given in Pyke (1984). Unlike the one-dimensional situation, multivariate KS statistics not only fail to be distribution free, but there is not even total agreement as to how they should be formulated. For example, should we consider the supremum of the difference between the empirical and hypothesised distribution functions, as in one dimension, or should we treat both of these as measures and consider the maximal difference between these measures as they vary over some class of sets (e.g. squares, disks, convex sets, etc.)?

Since for large sample sizes the empirical minus the hypothesised distribution function converges to the random field W_F defined in the preceeding section, any KS statistic reduces to studying the maximum of a Gaussian process. Thus, the methodology discussed above has impact here. For example, it is a straightforward calculation from the above described results of Samorodnitsky and myself to detail the form of the tail of the distribution of the KS statistics for large sample sizes. Furthermore, this calculation is not really any harder in the case of very complex parameter spaces, such as polygonal shapes or convex sets in R^2 as it is for the more classic, and simpler, parameter spaces. Examples are given in our three papers.

Further work was done on this and related problems during the year. In particular, while I was in Seattle Professor Ronald Pyke and I looked at the so-called "propagation of singularities" problem for random fields generated from empirical processes, and we have a number of preliminary results on this problem that we are still working on.

3(d) GENERALIZED PROCESSES

Let me start by defining two functions. If $X(u)$ is a Gaussian process, taking values in R^1 but with its parameter u in R^k, $k\geq 1$, then its *covariance function* $R(u,v)$ is defined by

$$R(u,v) = E\{X(u).X(v)\}.$$

If $W(t)$ is a Markov process defined on R^1_+, but taking values in R^k, and with stationary transition density $p_t(u,v)$, then its *Green's function* $g(u,v)$ is defined by

$$g(u,v) = \int_0^\infty e^{-t} p_t(u,v)\, dt.$$

It is a simple fact that every Green's function can serve as the covariance function of some (usually generalised) Gaussian field, and that the covariance functions of many Gaussian fields are also the Green's functions of Markov processes. This obvious, indeed, almost trite, fact has been known at least since the mid sixties, but had not been properly exploited until Dynkin, in a series of papers (1980 - 1984), used it to study Gaussian random fields from the viewpoint of Markov processes. Although Dynkin's approach was purely formal - i.e. it relied only on the fact that Green's functions and covariance functions were essentially equivalent objects - a simple physical bridge between the Markov and Gaussian situations also exists, and one of our suggestions in the proposal was to study this.

In fact, this was probably the most successful project of the last year. In a joint work with Ms. Raisa Epstein, who was fully supported under the grant as a research assistant, we wrote a very long and, we believe, useful account of the relation between Gaussian and non-Gaussian random fields and Markov processes. The link between them is via a central limit theorem for local times and other additive functionals of Markov processes, and is really too long to expand on here. Details can be found in Adler and Epstein (1986).

Work on various aspects of this project is still continuing. Part of it is related central (and other) limit theorems for additive functions of Brownian sheets, and some is related to a new, (and perhaps easier than the usual) treatment of problems related to the so-called "propagation of chaos" among weakly interacting systems of particles undergoing some Markov motion.

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1 Adler, R. J. and Epstein, R. (1986) A central limit theorem for Markov paths and some properties of Gaussian random fields.

2 Adler, R. J. and Samorodnitsky, G. (1985) Tail behaviour for the suprema of Gaussian processes with a view towards empirical processes. (Revision of earlier paper entitled "The supremum distribution of the Gaussian process".)

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4 Epstein, R. (1986) Limit theorems for additive functionals of the Brownian sheet. (In preparation.)

5 Samorodnitsky, G. (1986a) Bounds on the supremum distribution of Gaussian processes - polynomial entropy case.

6 Samorodnitsky, G. (1986b) Bounds on the supremum distribution of Gaussian processes - exponential entropy case.

6. CONFERENCES ATTENDED AND VISITS

I was fully supported by AFOSR during the months of July and August 1986. During that time I was at the following institutions, and spoke with the following mathematicians on a variety of problems related to the proposal:

July 1 - July 12	University of Washington; R. Pyke, K. Alexander
July 13 - July 20	Cornell University; N. U. Prabhu, E. B. Dynkin, R. Durrett.
July 21 - Aug. 5	University of Massachusetts at Amherst; J. Rosen, D. Geman, J. Horowitz, S. Ellis.
Aug. 6 - Aug. 10	Boston University and MIT; M. Taqqu, R. Dudley
Aug. 11 -Aug. 16	Attended conference "StatPhys 15", a trienniel conference on statistical physics. Boston.
Aug. 17 -Aug. 20	Clarkson University, Potsdam; L. Schulman.
Aug. 21 - Aug. 24	Carleton University, Ottowa; D. Dawson, M. Csorgo.

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